JOURNAL OF APPROXIMATION THEORY 77, 314–330 (1994)

Analytic Monotone Extension of Boundary Data

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Communicated by Ronald A. DeVore

Received September 12, 1992; accepted March 8, 1993

We derive several schemes that extend a monotone function on the boundary of the unit square to be a monotone function on the unit square while maintaining the smoothness of the function. Our results are related to those of Dahmen, DeVore, and Micchelli and have application to the modeling of charge distribution in semiconductor design. © 1994 Academic Press, Inc.

1. INTRODUCTION

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space, and let $\mathbb{R}^d_+$ denote the set of all points in $\mathbb{R}^d$ having nonnegative components. A function f is said to be monotone (nondecreasing) on a subset Ω of $\mathbb{R}^d$ if for any pair of $x, y \in \Omega$ with $x - y \in \mathbb{R}^d_+$, we have $f(x) \geqslant f(y)$. The monotonicity is said to be strict if $f(x) > f(y)$ whenever $x - y \in \mathbb{R}^d_+$ and $x \neq y$. Motivated by the modeling of charge distribution for semiconductor design, Dahmen, DeVore and Micchelli [1] studied the following problem:

Given a monotone function f on $\partial\Omega$, find a monotone function F on Ω such that

$$F(x) = f(x) \qquad \text{for all} \quad x \in \partial\Omega.$$

Such a function F is called a monotone extension of f. Dahmen, DeVore, and Micchelli [1] proved the following theorem:

THEOREM 1.1. *Let Ω be a bounded subset of $\mathbb{R}^d$ having nonempty interior. There is no linear and bounded operator L from $C(\partial\Omega)$ to $C(\Omega)$ such that Lf is monotone on Ω whenever f is monotone on $\partial\Omega$.*

In [1], several nonlinear methods of constructing monotone extensions were given. The case $\Omega = [0, 1]^2 \subset \mathbb{R}^2$ received particular attention. In this case, one starts with a function $f \in C([0, 1]^2)$ which is monotone on

0021-9045/94 $6.00

$\partial[0,1]^2$, and wishes to find an $F \in C([0,1]^2)$ which is monotone on $[0,1]^2$ and satisfies

$$F(x,0)=f(x,0), \qquad F(x,1)=f(x,1), \qquad 0 \leqslant x \leqslant 1,$$
$$F(0,y)=f(0,y), \qquad F(1,y)=f(1,y), \qquad 0 \leqslant y \leqslant 1.$$

It is practically important that F keep the smoothness of the boundary functions. This means the following: if $f(x,0)$, $f(x,1)$, $f(0,y)$, and $f(1,y)$ have certain degrees of smoothness as univariate functions, then one would like F to have the same degree of smoothness as a bivariate function. Furthermore, if the four boundary functions $f(x,0), f(x,1), f(0,y), f(1,y)$ are analytic on their domains, then one would like F to be an analytic bivariate function on $[0,1]^2$. We shall refer to such a function F as an *analytic monotone extension* (AME) of the given boundary data $f(x,0)$, $f(x,1), f(0,y), f(1,y)$.

To construct analytic monotone extensions, Dahmen, DeVore, and Micchelli considered the Boolean sum of the four boundary functions $f(x,0)$, $f(x,1)$, $f(0,y)$, $f(1,y)$, and by composing the Boolean sum with some carefully chosen univariate analytic functions, they successfully constructed AMEs for a class of boundary data that satisfy certain conditions.

The work of Dahmen, DeVore, and Micchelli naturally leads us to consider the following general construction scheme: For given boundary data $f(x,0), f(x,1), f(0,y), f(1,y)$, find a function G of eight variables analytic with respect to the first four variables, such that the function

$$F(x,y) := G(f(x,0), f(x,1), f(0,y), f(1,y),$$
$$f(0,0), f(1,0), f(0,1), f(1,1))$$

is a monotone extension of the boundary data. Ideally, we hope to find a sequence $\{G_n\}_{n=1}^{\infty}$ of such functions which achieve the following goal: For any given monotone boundary data $f(x,0), f(x,1), f(0,y), f(1,y)$, one can choose an element G_n of the sequence so that the function $F_n(x,y) :=$ $G_n(f(x,0),\ f(x,1),\ f(0,y),\ f(1,y),\ f(0,0),\ f(1,0),\ f(0,1),\ f(1,1))$ is a monotone extension of the boundary data.

This paper is organized as follows. In Section 2, we give several simple expressions of AMEs for some special classes of boundary functions. Our principal goal is to simplify the procedure established in [1] and to write AMEs in closed forms convenient for application. Our efforts to search for simpler expressions for AMEs are motivated by the practical importance of the problem and are inspired by the work of Dahmen, DeVore, and Micchelli. In Section 3, we show that all the current existing methods of constructing AMEs (including the one of Dahmen, DeVore, and Micchelli associated with Boolean sum) fail if extra conditions on the boundary data are not assumed. This implies that with the approach involved, the result of Theorem 4.1 in [1] cannot be improved. We also give some negative results on the solvability of analytic monotone extensions by a sequence of analytic functions. In Section 4, we discuss analytic monotone extension of higher orders.

2. Construction of Analytic Monotone Extensions in Special Cases

Let $\mathscr{M}([0,1]^2)$ denote the set of all functions in $C([0,1]^2)$ which are monotone on $\partial[0,1]^2$. Let $\mathscr{N}$ be a subset of $\mathscr{M}([0,1]^2)$, and $\mathscr{A} := \{G_n\}_{n \in \Lambda}$ be a set of at most countably many functions of eight variables, analytic with respect to the first four variables. We say that the problem of constructing AMEs for functions of $\mathscr{N}$ by $\mathscr{A}$ is solvable if the following scheme can be accomplished: for any $f \in \mathscr{N}$, one can find a $G_n \in \mathscr{A}$ so that the function

$$F(x, y) := G_n(f(x, 0), f(x, 1), f(0, y), f(1, y), \\ f(0, 0), f(1, 0), f(0, 1), f(1, 1))$$

is a monotone extension of the boundary data. In this case, we shall also say that $\mathscr{A}$ provides AMEs for functions of $\mathscr{N}$.

In this section, we investigate the solvability of this monotone extension scheme for some subsets of $\mathscr{M}([0,1]^2)$.

For convenience of discussion, we assume without loss of generality that our function $f \in C([0,1]^2)$ satisfies $f(0,0)=0$, $f(1,1)=1$. We also define $a := f(1,0)$ and $b := f(0,1)$. Our first result concerns the case $a=b$, and we shall use the symbol $\mathscr{M}_{a=b}$ to denote the set of all functions f in $\mathscr{M}([0,1]^2)$ satisfying $f(1,0)=f(0,1)$.

Proposition 2.1. *There exist three quadratic polynomials of four variables P_i, $i=1,2,3$, such that for any $f \in \mathscr{M}_{a=b}$, the function*

$$F(x, y) := P_i(f(x, 0), f(x, 1), f(0, y), f(1, y))$$

is a monotone extension of the boundary data for some i, $1 \leqslant i \leqslant 3$.

Proof. Consider the following three polynomials:

$$P_1(s, t, u, v) := tv,$$

$$P_2(s, t, u, v) := s + u - su,$$

$$P_1(s, t, u, v) := s + u - \frac{su}{a} + \frac{(t-a)(v-a)}{1-a}.$$

In the first case, assume $a = b = 0$. By the monotonicity of boundary data, we have $f(x, 0) \equiv 0$ and $f(0, y) \equiv 0$. It is easy to see that the function

$$F(x, y) := P_1(f(x, 0), f(x, 1), f(0, y), f(1, y)) = f(x, 1)\, f(1, y)$$

is a monotone extension of the boundary data. In the second case, assume $a = b = 1$. Then we have $f(x, 1) \equiv 1$ and $f(1, y) \equiv 1$, and we see that the function

$$\begin{aligned} F(x, y) &:= P_2(f(x, 0), f(x, 1), f(0, y), f(1, y)) \\ &= f(x, 0) + f(0, y) - f(x, 0)\, f(0, y) \end{aligned}$$

is a monotone extension of f.

Finally, assume $0 < a = b < 1$. Consider the function

$$\begin{aligned} F(x, y) &:= P_3(f(x, 0), f(x, 1), f(0, y), f(1, y)) \\ &= f(x, 0) + f(0, y) - \frac{f(x, 0)\, f(0, y)}{a} + \frac{[f(x, 1) - a][f(1, y) - a]}{1 - a} \end{aligned}$$

One easily verifies that F interpolates the boundary data. To see the monotonicity of F, let $x_1 > x_2$. We have

$$\begin{aligned} F(x_1, y) - F(x_2, y) := {} & [f(x_1, 0) - f(x_2, 0)]\left[1 - \frac{f(0, y)}{a}\right] \\ & + [f(x_1, 1) - f(x_2, 1)]\, \frac{[f(1, y) - a]}{1 - a}. \end{aligned}$$

Thus $F(x_1, y) - F(x_2, y) \geqslant 0$ follows from the assumption that the boundary data are monotone. Similarly, one shows that $F(x, y_1) - F(x, y_2) \geqslant 0$ under the condition $y_1 \geqslant y_2$. ∎

In what follows, we shall see that the violation of the seemingly indifferent condition $a = b$ brings essential difficulties in the construction of AME.

In handling the case $a \neq b$, we may assume, without loss of generality, that $a < b$, since otherwise we can switch the roles of the variables x and y. We may further assume that $0 < a < b < 1$. In fact, if $0 = a < b < 1$, then $f(x, 0) \equiv 0$, and the function

$$F(x, y) := \frac{1}{1 - b} \{f(1, y)\, [f(x, 1) - b] + f(0, y)\, [1 - f(x, 1)]\} \quad (2.1)$$

is an analytic monotone extension of f. If $a = 0$, $b = 1$, then $f(x, 0) \equiv 0$, $f(x, 1) \equiv 1$, and the function

$$F(x, y) := x f(1, y) + (1 - x)\, f(0, y) \quad (2.2)$$

is an analytic monotone extension of f. If $0<a<b=1$, then $f(x, 1) \equiv 1$, and the function

$$F(x, y):=\frac{1}{a}\{f(0, y)[a-f(x, 0)]+f(1, y) f(x, 0))\} \tag{2.3}$$

is an analytic monotone extension of f.

Let $\mathscr{M}_{a,b}$ denote the subset of all functions f in $\mathscr{M}([0,1]^2)$ satisfying $f(0,0)=0$, $f(1,0)=a$, $f(0,1)=b$, $f(1,1)=1$ with $0<a<b<1$. Let $\Omega_{a,b}$ denote the set defined by $\Omega_{a,b}:=\{(s,t,u,v): 0 \leqslant s \leqslant a,\ b \leqslant t \leqslant 1,\ 0 \leqslant u \leqslant b,\ a \leqslant v \leqslant 1,\ u \leqslant v\}$. In view of Eq. (2.1)–(2.3) and Proposition 2.1, we have the following result:

PROPOSITION 2.2. *Let $\mathscr{A}:=\{G_n\}$ be a set of at most countably many analytic functions of four variables on $\Omega_{a,b}$ which provides AMEs for functions of $\mathscr{M}_{a,b}$. Then there exists a set $\mathscr{A}_1:=\{G_n'\}$ of at most countably many functions of eight variables analytic with respect to the first four variables which provides AMEs for functions of $\mathscr{M}([0,1]^2)$.*

In what follows, we will concentrate on dealing with functions in $\mathscr{M}_{a,b}$.

LEMMA 2.3. *For any $f \in \mathscr{M}_{a,b}$, define an operator $L: \mathscr{M}_{a,b} \to \mathscr{M}_{a,b}$ by*

$$L(f, x, y):=\frac{f(x, 0) f(1, y)}{a}+\frac{f(x, 1) f(0, y)}{b}-\frac{f(x, 0) f(0, y)}{ab}. \tag{2.4}$$

Then Lf interpolates the boundary data $f(x,0)$, $f(x,1)$, $f(0,y)$, $f(1,y)$, and Lf is monotone on $[0,1]^2$ if the following inequalities hold true for all $0 \leqslant x$, $y \leqslant 1$

$$af(x, 1) \geqslant f(x, 0), \qquad bf(1, y) \geqslant f(0, y). \tag{2.5}$$

We omit the proof of Lemma 2.3 which only involves some straight forward calculations.

LEMMA 2.4. *Let $f \in \mathscr{M}_{a,b}$. Suppose that the boundary functions $f(x,0)$, $f(x,1)$, $f(0,y)$, $f(1,y)$ are differentiable and satisfy*

$$\begin{aligned} af_x'(x, 1) &\leqslant f_x'(x, 0), \qquad x \in [0,1], \\ bf_y'(1, y) &\leqslant f_y'(0, y), \qquad y \in [0,1]. \end{aligned} \tag{2.6}$$

Then the function $L(f, x, y)$ is an AME of f.

Proof. By the second part of Lemma 2.3, it suffices to show that $af(x,1) \geqslant f(x,0)$ and $bf(1,y) \geqslant f(0,y)$. Consider the function $I(x)=$

$af(x, 1) - f(x, 0)$. Then $I(1) = 0$, and for $0 \leqslant x \leqslant 1$, we have $I'(x) = af'_x(x, 1) - f'_x(x, 0) \leqslant 0$, since $af'_x(x, 1) \leqslant f'_x(x, 0)$. It follows that $I(x) \geqslant 0$ for all x, $0 \leqslant x \leqslant 1$. Similarly, we show that $bf(1, y) \geqslant f(0, y)$. ∎

Clearly, the conditions in Lemmas 2.3 and 2.4 will generally not be fulfilled. Influenced by the approach of Dahmen, DeVore, and Micchelli in the proof of Theorem 4.1, we wish to find a strictly increasing analytic function Φ_f: $[0, 1] \to [0, 1]$ of simple structure so that the function $\Phi_f \circ f$ satisfy the conditions of Lemma 2.4, i.e., we wish the following relations to hold for the function $\Phi_f \circ f$:

$$\begin{aligned} &\Phi_f(0) = 0, \ \Phi_f(1) = 1, \\ &[\Phi_f(f(x, 1))]'_x \leqslant [\Phi_f(f(x, 0))]'_x, \qquad x \in [0, 1], \\ &[\Phi_f(f(1, y))]'_y \leqslant [\Phi_f(f(0, y))]'_y, \qquad y \in [0, 1]. \end{aligned} \tag{2.7}$$

To this end, we consider the function

$$\Phi_n(t) := \frac{2^n - (t + 1/2)^{-n}}{2^n(1 - 1/3^n)}, \qquad n = 1, 2, \ldots .$$

It is easy to see that for each n, the function Φ_n is strictly increasing and analytic on $[0, 1]$ with $\Phi_n(0) = 0$, $\Phi_n(1) = 1$. Furthermore we have the following result:

Lemma 2.5. *Let $f \in \mathcal{M}_{a,b}$ be strictly monotone on $\partial[0, 1]^2$. Assume that the four functions $f'_x(x, 0)$, $f'_x(x, 1)$, $f'_y(0, y)$, $f'_y(1, y)$, are continuous and do not vanish on their domains. Then there exists a positive integer N, such that the following inequalities hold true for every Φ_n with $n \geqslant N$:*

$$\begin{aligned} &[\Phi_n(f(x, 1))]'_x \leqslant [\Phi_n(f(x, 0))]'_x, \qquad x \in [0, 1], \\ &[\Phi_n(f(1, y))]'_y \leqslant [\Phi_n(f(0, y))]'_y, \qquad y \in [0, 1]. \end{aligned} \tag{2.8}$$

Proof. We shall make use of the following three quantities derived from the boundary data $f(x, 0)$, $f(x, 1)$, $f(0, y)$, $f(1, y)$:

$$\begin{aligned} \delta &:= \min_{x, y} \{f(x, 1) - f(x, 0), f(1, y) - f(0, y)\}, \\ m &:= \min_{x, y} \{f'_x(x, 0), f'_x(x, 1), f'_y(0, y), f'_y(1, y)\}, \\ M &:= \max_{x, y} \{f'_x(x, 0), f'_x(x, 1), f'_y(0, y), f'_y(1, y)\}. \end{aligned} \tag{2.9}$$

It follows from the assumption on the boundary data that $\delta > 0$ and $m > 0$. We have

$$\frac{[\Phi_n(f(x,1))]'_x}{[\Phi_n(f(x,0))]'_x} = \frac{(f(x,1)+1/2)^{-(n+1)} f'_x(x,1)}{(f(x,0)+1/2)^{-(n+1)} f'_x(x,0)}$$
$$\leqslant \left[\frac{(f(x,0)+\delta+1/2)}{(f(x,0)+1/2)}\right]^{-(n+1)} \frac{M}{m}$$
$$\leqslant (1+2\delta)^{-(n+1)} \frac{M}{m}. \tag{2.10}$$

Hence there exists a natural number N such that the right-hand side of the above inequality is less than 1 for all $n > N$. Thus, we have proved the first inequality in (2.8). The second one follows similarly. ▮

We shall refer to the conditions of Lemma 2.5 as "Conditions $(*)$," and denote the set of all functions satisfying Conditions $(*)$ by $\mathcal{M}^*$.

THEOREM 2.6. *Let $f \in \mathcal{M}^*$ be fixed. Then there exists a natural number N so that the function $F_n(x, y) := \Phi_n^{-1} \circ L(\Phi_n \circ f, x, y)$ is an analytic monotone extension of f on $[0, 1]^2$ for each $n > N$, where Φ_n^{-1} denotes the inverse of Φ_n.*

Proof. For each $n = 1, 2, ...,$ we have $\Phi_n(f(0,0)) = 0$, $\Phi_n(f(1,1)) = 1$. If the boundary functions $f(x,0), f(x,1), f(0,y), f(1,y)$ satisfy the conditions of Lemma 2.5, then there exists a natural number N such that for each $n > N$, the functions $\Phi_n(f(x,0))$, $\Phi_n(f(x,1))$, $\Phi_n(f(0,y))$, $\Phi_n(f(1,y))$ satisfy the inequalities in (2.8). By Lemma 2.4, the function

$$L(\Phi_n \circ f, x, y) = \frac{\Phi_n(f(x,0))\,\Phi_n(f(1,y))}{\Phi_n(a)} + \frac{\Phi_n(f(x,1))\,\Phi_n(f(0,y))}{\Phi_n(b)}$$
$$- \frac{\Phi_n(f(x,0))\,\Phi_n(f(0,y))}{\Phi_n(a)\,\Phi_n(b)}$$

is an analytic monotone extension of the functions $\Phi_n(f(x,0))$, $\Phi_n(f(x,1))$, $\Phi_n(f(0,y))$, $\Phi_n(f(1,y))$. On the other hand, Φ_n^{-1} is also a strictly increasing analytic function on $[0, 1]$, and therefore for each $n > N$, the function $\Phi_n^{-1} \circ L(\Phi \circ f, x, y)$ is an analytic monotone extension of the boundary data $f(x,0), f(x,1), f(0,y), f(1,y)$. ▮

For purpose of comparison and for convenience of application, we give another construction.

LEMMA 2.7. *Let $\tilde{L}: \mathcal{M}_{a,b} \to \mathcal{M}_{a,b}$ be defined by*

$$\begin{aligned}\tilde{L}(f, x, y) = {} & \frac{1}{(1-a)(1-b)} f(x, 1) f(1, y) - \frac{1}{1-a} f(x, 0) f(1, y) \\ & - \frac{1}{1-b} f(x, 1) f(0, y) - \frac{b}{(1-a)(1-b)} f(1, y) \\ & - \frac{a}{(1-a)(1-b)} f(x, 1) + \frac{1}{1-a} f(x, 0) \\ & + \frac{1}{1-b} f(0, y) + \frac{ab}{(1-a)(1-b)}. \end{aligned} \tag{2.11}$$

Then $\tilde{L}$ interpolates the boundary data $f(x, 0)$, $f(x, 1)$, $f(0, y)$, $f(1, y)$. Furthermore, $\tilde{L}$ is an AME of f if f satisfies the following inequalities:

$$\begin{aligned} & f(x, 1) - (1-b) f(x, 0) - b \geqslant 0, \qquad x \in [0, 1], \\ & f(1, y) - (1-a) f(0, y) - a \geqslant 0, \qquad y \in [0, 1]. \end{aligned} \tag{2.12}$$

Proof. One directly verifies that $\tilde{L}(f, x, y)$ interpolates. To see the monotonicity of $\tilde{L}(f, x, y)$ with respect to x, let $x_1 \geqslant x_2$. We have

$$\begin{aligned} & \tilde{L}(f, x_1, y) - \tilde{L}(f, x_2, y) \\ & = \frac{1}{(1-b)} \left[\frac{f(1, y)}{(1-a)} - f(0, y) - \frac{a}{1-a} \right] [f(x_1, 1) - f(x_2, 1)] \\ & \quad + \frac{1}{1-a} [1 - f(1, y)] [f(x_1, 0) - f(x_2, 0)]. \end{aligned} \tag{2.13}$$

By the assumption $f(1, y) - (1-a) f(0, y) - a \geqslant 0$, $0 \leqslant y \leqslant 1$, we have $\tilde{L}(f, x_1, y) - \tilde{L}(f, x_2, y) \geqslant 0$ for all $0 \leqslant x_1 \leqslant x_2 \leqslant 1$ and $0 \leqslant y \leqslant 1$. Similarly, one verifies the monotonicity of $\tilde{L}(f, x, y)$ with respect to y. ∎

LEMMA 2.8. *$\tilde{L}(f, x, y)$ is an AME of f if f is monotone on $\partial[0, 1]^2$ and if the following inequalities hold true*:

$$\begin{aligned} & (1-b) f'_x(x, 0) \leqslant f'_x(x, 1), \qquad x \in [0, 1], \\ & (1-a) f'_y(0, y) \leqslant f'_y(1, y), \qquad y \in [0, 1]. \end{aligned} \tag{2.14}$$

Proof. Set $J(y) := (f(1, y)/(1-a)) - f(0, y) - a/(1-a)$. We see that $J(0) = 0$ and by the condition $(1-a) f'_y(0, y) \leqslant f'_y(1, y)$, we have $J'(y) \geqslant 0$ for all $0 \leqslant y \leqslant 1$. It follows that $J(y) \geqslant 0$ for all $y \in [0, 1]$. Therefore, from Equation (2.13), we have $\tilde{L}(f, x_1, y) - \tilde{L}(f, x_2, y) \geqslant 0$ for all $0 \leqslant x_1 \leqslant x_2 \leqslant 1$

and $0 \leqslant y \leqslant 1$. Similarly, we verify that $\tilde{L}(f, x, y_1) - \tilde{L}(f, x, y_2) \geqslant 0$ for all $0 \leqslant y_1 \leqslant y_2 \leqslant 1$ and $0 \leqslant x \leqslant 1$. ∎

We notice that the conditions of Lemma 2.7 can only be satisfied by a small class of boundary functions. Again we borrow the idea of Dahmen, DeVore, and Micchelli in the proof of Theorem 4.1 [1]. For a given function $f \in C([0, 1]^2)$ we wish to find a function $\tilde{\Phi}_f$ so that the function $\tilde{\Phi}_f \circ f$ satisfy the conditions of Lemma 2.7. For this purpose, we use the functions $\tilde{\Phi}_n$

$$\tilde{\Phi}_n := \frac{(t+1)^n - 1}{2^n - 1}, \qquad n = 1, 2, \ldots .$$

The following theorem then follows in a way similar to Theorem 2.6.

THEOREM 2.9. *Let $f \in \mathcal{M}^*$ be fixed. There exists a natural number N so that the function*

$$\tilde{F}_n(x, y) := \tilde{\Phi}_n^{-1} \bullet L(\tilde{\Phi}_n \circ f, x, y)$$

is an analytic monotone extension of f on $[0, 1]^2$ for each $n > N$, where $\tilde{\Phi}_n^{-1}$ denotes the inverse of $\tilde{\Phi}_n$.

There is an intimate relationship between the operators L, $\tilde{L}$ and the Boolean sum operator L_1 of Dahmen, DeVore, and Micchelli defined by

$$\begin{aligned} L_1(f, x, y) := {} & \phi(x) f(0, y) + (1 - \phi(x)) f(1, y) \\ & + \psi(y) f(x, 0) + (1 - \psi(y)) f(x, 1) \\ & - \{ f(0, 0) \phi(x) \psi(y) + f(0, 1) \phi(x)(1 - \psi(y)) \\ & + f(1, 0)(1 - \phi(x)) \psi(y) + f(1, 1)(1 - \phi(x))(1 - \psi(y)) \}. \end{aligned} \tag{2.15}$$

Here the functions $\phi(x)$, $\psi(y)$ are defined by

$$\begin{aligned} \phi(x) &:= \frac{f(x, 1) - f(x, 0) - f(1, 1) + f(1, 0)}{\Delta}, \\ \psi(y) &:= \frac{f(1, y) - f(0, y) - f(1, 1) + f(0, 1)}{\Delta}, \end{aligned} \tag{2.16}$$

where $\Delta := -f(0, 0) - f(1, 1) + f(0, 1) + f(1, 0)$ is assumed not be zero.

Dahmen, DeVore, and Micchelli [1] showed that $L_1(f, x, y)$ is an AME of f if f is monotone on $\partial[0, 1]^2$ and if the following relations hold:

$$\begin{aligned} f(x, 1) - f(x, 0) &\in [f(1, 1) - f(1, 0), f(0, 1) - f(0, 0)], \\ f(1, y) - f(0, y) &\in [f(1, 1) - f(0, 1), f(1, 0) - f(0, 0)]. \end{aligned} \tag{2.17}$$

We have the following observation:

PROPOSITION 2.10. *If the relations in* (2.17) *hold true, then either the ones in* (2.5) *or the ones in* (2.12) *hold true.*

Proof. We recall that $f(0,0)=0$, $f(1,0)=a$, $f(0,1)=b$ and $f(1,1)=1$. Suppose the relations in (2.17) hold true and $\Delta>0$. We have

$$\begin{aligned} af(x,1)-f(x,0) &= [f(x,1)-f(x,0)]-f(x,1)(1-a) \\ &\geqslant (1-a)-f(x,1)(1-a)\geqslant 0, \\ bf(1,y)-f(0,y) &= [f(1,y)-f(0,y)]-f(1,y)(1-b) \\ &\geqslant (1-b)-f(1,y)(1-b)\geqslant 0. \end{aligned}$$

Suppose the relations in (2.17) hold true and $\Delta<0$. We have

$$\begin{aligned} &f(x,1)-(1-b)\,f(x,0)-b \\ &\quad=[f(x,1)-f(x,0)]+bf(x,0)-b\geqslant bf(x,0)\geqslant 0, \\ &f(1,y)-(1-a)\,f(0,y)-a \\ &\quad=[f(1,y)-f(0,y)]+af(0,y)-a\geqslant bf(x,0)\geqslant 0. \quad\blacksquare \end{aligned}$$

The above discussions yields the folllowing result:

THEOREM 2.11. *There exists a sequence* $\{G_n\}_{n=1}^{\infty}$ *of analytic functions of four variables which provides analytic monotone extensions to functions of* $\mathcal{M}^*$.

Although Theorems 2.6 and 2.9 give some different and simple constructions of analytic monotone extensions, the approaches involved are essentially in the same vein as those of Dahmen, DeVore, and Micchelli [1] in the proof of their Theorem 4.1. All of these construction methods require the fulfillment of the extra conditions that the function f is strictly monotone on $\partial[0,1]^2$ and the four boundary functions $f(x,0)$, $f(x,1)$, $f(0,y)$, $f(1,y)$ are differentiable and their derivatives do not vanish on their domains. In the next section, we shall show that with the approach involved, these conditions are necessary.

3. FURTHER INVESTIGATIONS

THEOREM 3.1. *Let* $G\in C(\Omega_{a,b})$. *Assume that* G *satisfies the following equations*:

$$\begin{aligned} G(0,b,u,v)&=u, \qquad G(a,1,u,v)=v, \\ G(s,t,0,a)&=s, \qquad G(s,t,b,1)=t. \end{aligned} \tag{3.1}$$

Define an operator $\mathscr{T}: \mathscr{M}_{a,b} \to \mathscr{M}_{a,b}$ *by* $\mathscr{T}: f \mapsto F$, *where*

$$F(x, y) := G(f(x, 0), f(x, 1), f(0, y), f(1, y)).$$

Then $\mathscr{T}f$ *interpolates the boundary data* $f(x, 0)$, $f(x, 1)$, $f(0, y)$, $f(1, y)$. *Furthermore,* $\mathscr{T}f$ *is monotone on* $[0, 1]^2$ *for every* $f \in \mathscr{M}_{a,b}$ *if and only if* G *is monotone on* $\Omega_{a,b}$.

Proof. It is obvious that $\mathscr{T}f$ interpolates f on the boundary if G satisfies (3.1). It is also obvious that $\mathscr{T}f$ is monotone on $[0, 1]^2$ for every $f \in \mathscr{M}_{a,b}$ if G is monotone on $\Omega_{a,b}$. To show that the condition G is monotone on $\Omega_{a,b}$ is also necessary, assume that G is not monotone on $\Omega_{a,b}$. Then, there exist two points $W_1 = (s_1, t_1, u_1, v_1)$, $W_2 = (s_2, t_2, u_2, v_2)$ in $\Omega_{a,b}$, such that $W_1 - W_2 \in \mathbb{R}^4_+$ and $G(W_1) - G(W_2) < 0$. In this case, we can find two points $X_1 := (x_1, y_1)$, $X_2 := (x_2, y_2)$ in $[0, 1]^2$ and a function f in $\mathscr{M}_{a,b}$ such that $X_1 - X_2 \in \mathbb{R}^2_+$ and

$$\begin{array}{llll} f(x_1, 0) = s_1, & f(x_1, 1) = t_1, & f(0, y_1) = u_1, & f(1, y_1) = v_1, \\ f(x_2, 0) = s_2, & f(x_2, 1) = t_2, & f(0, y_2) = u_2, & f(1, y_2) = v_2. \end{array}$$

We thus have $F(x_1, y_1) - F(x_2, y_2) = G(s_1, t_1, u_1, v_1) - G(s_2, t_2, u_2, v_2) < 0$. This means that $\mathscr{T}f$ is not monotone on $[0, 1]^2$ for this particular f and leads to a contradiction. ▮

Theorem 3.1 enables us to change the problem of finding monotone extensions of functions in $\mathscr{M}_{a,b}$ to a simpler problem of finding a single continuous monotone function G on $\Omega_{a,b}$ satisfying (3.1). It is easily seen that such functions exist. In working with AMEs, we need an analytic monotone function G on $\Omega_{a,b}$ that satisfies (3.1). At this stage, we are not certain about the existence of such an analytic function. However, with little effort, we can show that it is impossible to find such a G from a certain subset of analytic functions. Let $\mathscr{B}$ denote the set of all functions G analytic on the poly-disc $D := \{(s, t, u, v) \colon |s| \leqslant a,\ |t| \leqslant 1,\ |u| \leqslant b,\ |v| \leqslant 1\}$ having an expansion of the form

$$G(s, t, u, v) = \sum_{n=0}^{\infty} \sum_{\mu + \nu = n} P_{\mu,\nu}(s, t)\, u^{\mu} v^{\nu},$$

where $P_{\mu,\nu}(s, t)$ are analytic functions on the disc $\{|s| \leqslant a, |t| \leqslant 1\}$, and for each $n \geqslant 2$, all but one $P_{\mu,\nu}(s, t)$, $\mu + \nu = n$, are identically zero.

PROPOSITION 3.2. *There is no function in* $\mathscr{B}$ *that satisfies* (3.1) *and at the same time is monotone on* $\Omega_{a,b}$.

Proof. We will prove by contradiction. Assume that $G \in \mathscr{B}$ satisfying (3.1) and is monotone on $\Omega_{a,b}$. For any fixed α, $a<\alpha<b$, we have $G(0, b, \alpha, \alpha)=\alpha$ and $G(a, 1, \alpha, \alpha)=\alpha$ by (3.1). Since G is monotone on $\Omega_{a,b}$, $G(s, t, \alpha, \alpha)=\alpha$ for all s and t with $(s, t, \alpha, \alpha) \in \Omega_{a,b}$. Hence we have

$$\alpha = \sum_{n=0}^{\infty} \left\{ \sum_{\mu+\nu=n} P_{\mu,\nu}(s, t) \right\} \alpha^n. \tag{3.2}$$

Since for each n, $n \geqslant 2$, all but one $P_{\mu,\nu}(s, t)$, $\mu+\nu=n$ are identically zero, Eq. (3.2) shows that

$$P_{0,0}(s, t)=0, \qquad P_{1,0}(s, t)+P_{0,1}(s, t)=1,$$
$$P_{\mu,\nu}(s, t)=0, \qquad \mu+\nu=n \geqslant 2.$$

Therefore, we have $G(s, t, u, v)=P_{1,0}(s, t)\, u+(1-P_{1,0}(s, t))\, v$. The condition $G(s, t, 0, a)=s$ implies that $P_{1,0}(s, t)=1-s/a$. However, the condition $G(s, t, b, 1)=t$ implies that $P_{1,0}(s, t)=(t-1)/(b-1)$. This is a contradiction. ∎

PROPOSITION 3.3. *There is no quadratic polynomial of the four variables* s, t, u, v *that satisfies* (3.1) *and at the same time is monotone on* $\Omega_{a,b}$.

Proof. Assume that $P(s, t, u, v)$ is such a quadratic polynomial. We write

$$P(s, t, u, v)=P_1(s, t, u, v)+a_1 u^2+a_2 v^2+a_3 uv,$$

where a_1, a_2, a_3 are some constants, and P_1 is a certain quadratic polynomial not containing the terms u^2, v^2 and uv. Since P satisfies (3.1), we have $a_1=a_2=a_3=0$. This means that P_1 satisfies (3.1) and is monotone on $\Omega_{a,b}$. This contradicts Proposition 3.2 since $P_1 \in \mathscr{B}$. ∎

Let P be the polynomial of four variables defined by

$$P(s, t, u, v)=\frac{sv}{a}+\frac{tv}{b}-\frac{su}{ab}.$$

By Lemma 2.3, P satisfies (3.1), however P is not monotone on $\Omega_{a,b}$. If $f \in \mathscr{M}_{a,b}$ satisfies (2.5), then by Lemma 2.3, the function $F(x, y):= P(f(x, 0), f(x, 1), f(0, y), f(1, y))$ is an AME of f. Due to the stringency of the conditions in (2.5), we resort to the sequence $\{\Phi_n\}$ defined in Section 2, and hope that for each fixed $f \in \mathscr{M}_{a,b}$, the function $\Phi_n \circ f$ satisfies (2.5) when n is sufficiently large. Theorem 2.6 ensures that this method works for functions satisfying Conditions (*). With the techniques involved, the result of Theorem 2.6 is the best we can expect, Conditions (*) are all essential for the construction scheme to be successful. If we do not require that f be strictly monotone on $\partial[0, 1]^2$, then we can find a number α, $a<\alpha<b$, and a function $f \in \mathcal{M}_{a,b}$ such that $f(0, 1/2)=f(1, 1/2)=\alpha$. For each n, in view of (2.4), we have

$$\left.\frac{\partial L(\Phi_n \circ f, x, y)}{\partial x}\right|_{y=1/2}=\frac{1}{\Phi_n(a)}\left(\Phi_n(\alpha)-\frac{\Phi_n(\alpha)}{\Phi_n(b)}\right)\Phi_n'(f(x, 0))\, f_x'(x, 0) + \frac{\Phi_n(\alpha)}{\Phi_n(b)}\Phi_n'(f(x, 1))\, f_x'(x, 1). \tag{3.3}$$

Since Φ_n' are decreasing functions, we have $\Phi_n'(f(x, 1))<\Phi_n'(f(x, 0))$. Thus we can carefully design the two functions $f_x'(x, 0)$, $f_x'(x, 1)$ such that for each n, the function $\partial L(\Phi_n \circ f, x, y)/\partial x$ is negative at the point $(x, 1/2)$ when x is close to 1 even the derivatives of $f(x, 0)$, $f(x, 1)$, $f(0, y)$, $f(1, y)$ are all positive. Therefore for each n, the function $L(\Phi_n \circ f, x, y)$ is not monotone on $[0, 1]^2$.

Now we assume that f is strictly monotone on $\partial[0, 1]^2$ but allow some of the derivatives of $f(x, 0)$, $f(x, 1)$, $f(0, y)$, $f(1, y)$ to have isolated zeros. Suppose that $f_y'(1, y_0)=0$, for some $0<y_0<1$. Although we have $\Phi_n(f(0, y))<\Phi_n(f(1, y))$, for all $y \in [0, 1]$, we can make $f'(1, y)$ and $f(1, y)-f(0, y)$ so small in a neighborhood of y_0 that for each n, the function $\Phi_n(f(1, y))-(\Phi_n(f(0, y))/\Phi_n(b))$ is negative on some neighborhood of y_0. By the equation

$$\frac{\partial L(\Phi_n \circ f, x, y)}{\partial x}=\frac{1}{\Phi_n(a)}\left(\Phi_n(f(1, y))-\frac{\Phi_n(f(0, y))}{\Phi_n(b)}\right) \times \Phi_n'(f(x, 0))\, f_x'(x, 0) + \frac{\Phi_n(f(0, y))}{\Phi_n(b)}\Phi_n'(f(x, 1))\, f_x'(x, 1), \tag{3.4}$$

we can carefully design $f_x'(x, 0)$, $f_x'(x, 1)$ so that for each n, the function $\partial L(\Phi_n \circ f, x, y)/\partial x$ is negative somewhere in $[0, 1]^2$, and therefore for each n, the function $L(\Phi_n \circ f, x, y)$ is not monotone on $[0, 1]^2$.

We have also analyzed the operator $\tilde{L}$ defined in (2.11) and the Boolean sum operator L_1 by Dahmen, DeVore, and Micchelli as in (2.15), and found that they too suffer from the limitation of having to require functions to satisfy Conditions (*). The following theorem also casts some shadow on the construction of AMEs.

THEOREM 3.4. *Let $\mathscr{A} := \{G_n\}_{n=1}^{\infty}$ be a sequence of analytic functions of four variables on the set $\Omega_{a,b}$. Let $[a_1, b_1] \subset (a, b)$. Assume that for each $\alpha \in [a_1, b_1]$, and each n, there is a neighborhood $N_{\alpha,n}$ of the point $(a, 1, \alpha, \alpha)$*

restricted to the subspace topology on the set $T_\alpha := \{(s, t, u, \alpha) : (s, t, u, \alpha) \in \mathcal{O}\}$, *where* $\mathcal{O}$ *is an open set containing* $\Omega_{a,b}$ *where each* G_n *is analytic. Assume that the three partials* $\partial G_n/\partial s$, $\partial G_n/\partial t$, $\partial G_n/\partial u$ *are nonnegative on* $N_{\alpha,n}$. *Then monotone extensions for functions of* $\mathcal{M}_{a,b}$ by $\mathcal{A}$ *is not solvable.*

Proof. We shall prove this theorem by contradiction. Assume that $\mathcal{A}$ provides AMEs for functions of $\mathcal{M}_{a,b}$. Without loss of generality, we may assume that $\mathcal{A}$ is *minimum*, which means that if any subset Q of $\mathcal{A}$ provides AMEs for functions of $\mathcal{M}_{a,b}$, then $Q = \mathcal{A}$. Let α, $a_1 < \alpha < b_1$ be fixed. Let $\mathcal{M}_\alpha$ denote the subset of all functions f in $\mathcal{M}_{a,b}$ satisfying the following conditions: $f(0, 1/2) = f(1, 1/2) = \alpha$, $a_1 < \alpha < b_1$, $f(1, y) = \alpha$, $1/4 \leqslant y \leqslant 1/2$. For every $f \in \mathcal{M}_\alpha$, there exists $n > 0$, depending on f, such that the function $G_n(f(x, 0), f(x, 1), f(0, y), f(1, y))$ is a monotone extension of f. Thus we have

$$\begin{aligned} G_n(f(x, 0), f(x, 1), \alpha, \alpha) &= \alpha, \\ G_n(a, 1, f(0, y), \alpha) &= \alpha, \qquad 1/4 \leqslant y \leqslant 1/2. \end{aligned} \tag{3.5}$$

Since in (3.5), the functions $f(x, 0)$, $f(x, 1)$, $f(0, y)$ may vary independently as long as the monotonicity of f on $\partial[0, 1]^2$ be maintained, we observe that the set A,

$$A := \{(s, t, u) : \exists n, \text{ such that } G_n(s, t, \alpha, \alpha) = \alpha \text{ and } G_n(a, 1, u, \alpha) = \alpha\},$$

as a subset of $\mathbb{R}^3$ has positive measure. Therefore, there exists a natural number n_0 such that the set A_0,

$$A_0 =: \{(s, t) : G_{n_0}(s, t, \alpha, \alpha) = \alpha\},$$

and the set B_0,

$$B_0 := \{u : G_{n_0}(a, 1, s, \alpha) = \alpha\},$$

both have positive measure as subsets of $\mathbb{R}^2$ and $\mathbb{R}$, respectively. The function $G_{n_0}(s, t, \alpha, \alpha)$, for fixed α, is analytic with respect to the two variables s, t. Hence $G_{n_0}(s, t, \alpha, \alpha) = \alpha$ for all s and t with $(s, t, \alpha, \alpha) \in \Omega_{a,b}$. For the same reason $G_{n_0}(a, 1, u, \alpha) = \alpha$ for all u with $(a, 1, u, \alpha) \in \Omega_{a,b}$. Since $\partial G_n/\partial s$, $\partial G_n/\partial t$, $\partial G_n/\partial u$, are nonnegative in a neighborhood of $(a, 1, \alpha, \alpha)$ with respect to the subspace topology on the set T_α, we can find an open set C of $\mathbb{R}^3$, such that

$$G_{n_0}(s, t, u, \alpha) = \alpha, \qquad (s, t, u) \in C.$$

Applying the analyticity argument again, we conclude that

$$G_{n_0}(s, t, u, \alpha) = \alpha, \qquad (s, t, u, \alpha) \in \Omega_{a,b}. \tag{3.6}$$

We have proved that for each fixed α, $a<\alpha<b$, there exists a natural number n_0 such that Eq. (3.6) is true. The cardinality argument yields that there exists a natural number n_1 and a subset E of (a_1, b_1) of positive measure such that

$$G_{n_1}(s, t, u, \alpha)=\alpha, \qquad \text{whenever} \quad (s, t, u, \alpha)\in \Omega_{a,b} \text{ and } \alpha\in E.$$

The analyticity of G_{n_1} on the set $\Omega_{a,b}$ implies that $G_{n_1}(s, t, u, v)=v$ for all $(s, t, u, v)\in \Omega_{a,b}$. It is easy to see that for any $f\in \mathscr{M}_{a,b}$, the function

$$G_{n_1}(f(x, 0), f(x, 1), f(0, y), f(1, y))$$

does not interpolate the boundary functions $f(x, 0), f(x, 1), f(0, y), f(1, y)$. This contradicts the assumption that $\mathscr{A}$ is minimum. ▮

4. Higher Degree of Monotonicity

Let k be a natural number, and let $h_j\in \mathbb{R}^d_+$ $(j=1, ..., k)$. For $f\in C(\mathbb{R}^d)$, we define

$$\Delta^1 f_{h_1}(\cdot):=\Delta f_{h_1}(\cdot):=f(\cdot+h_1)-f(\cdot),$$
$$\Delta^2 f_{h_1h_2}(\cdot):=\Delta(\Delta f_{h_1})_{h_2}(\cdot),$$

and iterately

$$\Delta^k f_{h_1h_2\cdots h_k}(\cdot):=\Delta(\Delta^{k-1} f_{h_1h_2\cdots h_{k-1}})_{h_k}(\cdot).$$

Definition 4.1. Let $\Omega\subset \mathbb{R}^d$. A function f defined on Ω is said to be k-monotone (increasing) on Ω if $\Delta^k f_{h_1h_2\cdots h_k}(\cdot)\geqslant 0$ whenever the points involved are in Ω.

Let Ω be a connected region in $\mathbb{R}^d$, let f be kth differentiable on Ω. Then, f is k-monotone on Ω if and only if $D^\beta f(x)\geqslant 0$ for all $x\in\Omega$ and all multi-indices β with $|\beta|=k$.

It is natural to study the following problem: for a given function f being k-monotone on $\partial[0, 1]^2$, find a k-monotone function F on $[0, 1]^2$ that agrees with f on the $\partial[0, 1]^2$. Such a function F is called a k-monotone extension of the function f. The following theorem gives a partial solution to this problem for the case $k=2$.

Theorem 4.2. *Let $f\in C([0, 1]^2)$. Suppose that the four boundary functions $f(x, 0), f(x, 1), f(0, y), f(1, y)$ have nonnegative second derivatives and the following inequalities hold true*:

$$f'_x(x, 1)\geqslant f'_x(x, 0),\ 0\leqslant x\leqslant 1, \qquad f'_y(y, 1)\geqslant f'_y(y, 0),\ 0\leqslant y\leqslant 1. \quad (4.1)$$

Then for any order of smoothness possessed by the four boundary functions $f(x, 0)$, $f(x, 1)$, $f(0, y)$, $f(1, y)$, *there exists a 2-monotone extension* F *of* f *possessing the same order of smoothness on* $[0, 1]^2$. *Moreover, if the four boundary functions*

$$f(x, 0), f(x, 1), f(0, y), f(1, y)$$

are analytic as univariate functions on their domains, then the function F *is a bivariate analytic function on* $[0, 1]^2$.

Proof. Recall Eq. (2.15) in which the Boolean sum operator used in [1] is defined with the functions $\phi(x)$, $\psi(y)$ being at our disposal. We borrow from [1] the following equations:

$$\frac{\partial L_1}{\partial x}(x, y) = \psi(y) f'_x(x, 0) + (1 - \psi(y)) f'_x(x, 1) + \phi'(x) \times [\Delta\psi(y) + f(0, y) - f(1, y) - f(0, 1) + f(1, 1)], \quad (4.2)$$

$$\frac{\partial L_1}{\partial y}(x, y) = \phi(x) f'_y(0, y) + (1 - \phi(x)) f'_y(y, 1) + \psi'(y) \times [\Delta\phi(x) + f(x, 0) - f(x, 1) - f(1, 0) + f(1, 1)]. \quad (4.3)$$

We first consider the case $\Delta = 0$. In this case, we choose $\phi(x) = (1 - x)^2$, $\psi(y) = (1 - y)^2$. Then $\phi(0) = \psi(0) = 1$ and $\phi(1) = \psi(1) = 0$. In view of the pertinent discussion on [1], such choices of the functions ϕ and ψ guarantee that L_1 interpolates f on $\partial[0, 1]^2$. Besides, from Eqs. (4.2) and (4.3), we obtain

$$\frac{\partial^2 L_1}{\partial x^2}(x, y) = \psi(y) f''_x(x, 0) + (1 - \psi(y)) f''_x(x, 1) + \phi''(x) \times [f(0, y) - f(1, y) - f(0, 1) + f(1, 1)], \quad (4.4)$$

$$\frac{\partial^2 L_1}{\partial y^2}(x, y) = \phi(x) f''_y(0, y) + (1 - \phi(x)) f''_y(y, 1) + \psi''(y) \times [f(x, 0) - f(x, 1) - f(1, 0) + f(1, 1)], \quad (4.5)$$

and

$$\frac{\partial^2 L_1}{\partial x\, \partial y}(x, y) = \psi'(y)(f'_x(x, 0) - f'_x(x, 1)) + \phi'(x)(f'_y(0, y) - f'_y(1, y)). \quad (4.6)$$

By the Mean-Value Theorem and (4.1), we have

$$\begin{aligned} f(0, y) - f(1, y) - f(0, 1) + f(1, 1) &= (y - 1)(f'_y(0, \xi) - f'_y(1, \xi)) \geqslant 0, \\ f(x, 0) - f(x, 1) - f(1, 0) + f(1, 1) &= (x - 1)(f'_x(\eta, 0) - f'_x(\eta, 1)) \geqslant 0, \end{aligned} \quad (4.7)$$

where ξ, η satisfy $y \leqslant \xi \leqslant 1$ and $x \leqslant \eta \leqslant 1$. Noticing $0 \leqslant \phi(x) \leqslant 1$, $0 \leqslant \psi(y) \leqslant 1$, $\phi'(x) \leqslant 0$, $\psi'(y) \leqslant 0$, and $\phi''(x) = \psi''(y) = 2$, and in view of Eqs. (4.1), (4.4), and (4.5)–(4.7), we have for all $(x, y) \in [0, 1]^2$ that

$$\frac{\partial^2 L_1}{\partial x^2}(x, y) \geqslant 0, \qquad \frac{\partial^2 L_1}{\partial y^2}(x, y) \geqslant 0, \qquad \frac{\partial^2 L_1}{\partial x\, \partial y}(x, y) \geqslant 0. \tag{4.8}$$

Hence L_1 is an extension of f satisfying our requirements.

Now assume $\Delta \neq 0$. In this case, we let the functions $\phi(x)$, $\psi(y)$ be as defined in Eq. (2.16). With these choices of the functions $\phi(x)$, $\psi(y)$, $L_1(x, y)$ interpolates f on $\partial[0, 1]^2$ and meets the smoothness requirement. We have

$$\begin{aligned} \frac{\partial^2 L_1}{\partial x^2}(x, y) &= \psi(y) f''_x(x, 0) + (1 - \psi(y)) f''_x(x, 1), \\ \frac{\partial^2 L_1}{\partial y^2}(x, y) &= \phi(x) f''_y(0, y) + (1 - \phi(x)) f''_y(y, 1), \\ \frac{\partial^2 L_1}{\partial x\, \partial y}(x, y) &= \psi'(y)(f'_x(x, 0) - f'_x(x, 1)) \\ &= \frac{1}{\Delta}[f'_y(1, y) - f'_y(0, y)][f'_x(x, 0) - f'_x(x, 1)]. \end{aligned} \tag{4.9}$$

From Equations in (4.7) and the definition of Δ, we have $\Delta < 0$ and $\phi(x) \geqslant 0$, $\psi(y) \geqslant 0$. Similarly, applying The Mean-Value Theorem and the inequalities in (4.1), we have

$$\begin{aligned} f(1, y) - f(0, y) - f(1, 0) + f(0, 0) \geqslant 0, \\ f(1, y) - f(0, y) - f(1, 0) + f(0, 0) \geqslant 0, \end{aligned} \tag{4.10}$$

Equation (4.10) implies that $\phi(x) \leqslant 1$, $\psi(y) \leqslant 1$. It then follows from Eqs. (4.1) and (4.9) that for all $(x, y) \in [0, 1]^2$,

$$\frac{\partial^2 L_1}{\partial x^2}(x, y) \geqslant 0, \qquad \frac{\partial^2 L_1}{\partial y^2}(x, y) \geqslant 0, \qquad \frac{\partial^2 L_1}{\partial x\, \partial y}(x, y) \geqslant 0. \quad \blacksquare$$

Acknowledgment

We thank Professor R. DeVore for his advice that has enhanced the quality of this paper.

Reference

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